Sequential Vessel Speed Optimization under Dynamic Weather Conditions

Debabrota Basu*, Giulia Pedrielli†, Weidong Chen*, Szu Hui Ng‡, Loo Hay Lee‡, Stéphane Bressan*
*School of Computing, National University of Singapore, Singapore.
†School of Computing, Informatics and Design Systems Engineering, Arizona State University, USA.
‡Department of Industrial and Systems Engineering, National University of Singapore, Singapore.

***Abstract*—The International Maritime Organization (IMO) identifies speed optimization as a key operational measure for achieving energy efficiency through reduced emissions. Ocean Liner services have fixed port rotations and schedules. While the speed can be optimized for emissions, the service level in terms of scheduled arrival and departure need to be carefully considered not to loose market share. This already challenging problem is further complicated when dynamic weather conditions along the service route are considered. In fact, few contributions can be found that address this issue.**

We study the operational problem of dynamically determining a vessel's speed, departure time and arrival time at each port of call under dynamic weather conditions. We model the minimization of cost, namely bunkering costs and early and delayed departure and arrival penalties, using the calculus of variations. The proposed algorithm leverages upon a discretization technique based on the Weierstrass–Erdmann condition. The numerical tests show the efficiency and effectiveness of this algorithm over standard techniques like IVP.

***Index Terms*—Vessel speed optimization, Calculus of variations, Weierstrass-Erdmann condition.**

I. Background and Contribution

International shipping contributes to Global Greenhouse Gas (GHG) emissions. In 2007, approximately 2.7% (i.e. 870 million tons) of global CO_2 emissions were attributed to shipping [1]. As the reliance of the world economy on the global trade of commodities and manufactured products continues to increase, the emission of CO_2 by shipping is expected to rise to between 2500 and 3650 million tons by 2050 [1]. Consequently, the International Maritime Organization (IMO) is working on introducing practices and regulations to reduce greenhouse gas emission by the shipping industry.

Moreover, bunkering costs constitute nearly three-quarters of the total operating costs for a large container ship [2]. In recent years, bunker prices have considerably raised. An increase in the bunker fuel price has an upward effect on costs. Whether or not shipping companies pass the costs on to the customer through variable charges - and this may not be an option due to the prevailing oversupply of vessels and slowdown in the global economy [3] - such an increase has a negative impact on the shipping industry and the global economy. Hence, shipping lines and other stakeholders are challenged to keep a tighter control on bunker fuel consumption.

Ocean liner services have a fixed port rotation and operate on a published schedule. The vessel operating the specific service calls at advertised dates and times at ports, which are specified by the company ahead of time (usually the port rotation does not change throughout the year). As a result, the operator of the vessel wants to reduce bunker consumption while respecting the published schedule in order not to incur in loss of market shares.

TABLE I: Notations

Parameters	
n	number of port of calls.
d_i	distance between the first port of call and port of call i along the route.
D	total distance of the voyage, which is the same as d_n.
T	the maximum time we will considered.
V_{min}	minimum speed of the vessel.
V_{max}	maximum speed of the vessel.
$[t'_i, t''_i]$	soft arrival time window at port of call i.
$Q_i(t)$	early/late arrival penalty cost.
e_i	port time (including entry time, unloading time,loading time, idle time and exit time) at port i.
c_i	bunker price at port i.
$\phi(x,t)$	a function to present the impact of weather condition at position x and time t.
$F(\phi(x,t),v)$	the instantaneous bunker consumption rate.
$L(t,v,x)$	equals to $F(\phi(x,t),v)$.
$\{D^p\}_{0\le p\le N_d}$,	discretized distance with $0 = D^0 \le D^1 \le \cdots \le D^{N_d} = D$.
$\{T^q\}_{0\le p\le N_t}$,	discretized time with $0 = T^0 \le T^1 \le \cdots \le T^{N_t} = T$.
$R_{p,q}$	denotes rectangle $[D^p, D^{p+1}) \times [T^q, T^{q+1})$.
$L_{p,q}(v), \hat{L}_{p,q}(v)$	the instantaneous bunker consumption rate inside $R_{p,q}$.
Dependent variables	
$x(t)$	the distance between the ship and the first port of call along the route at time instance t.
t_i	arrival time at port i.
$\bar{t}_i$	departure time at port i, which equals to $t_i + e_i$.
v_i	departure speed at port i.
Decision variable	
$v(t)$	speed of the vessel at time t along the route.

This problem is not trivial as the bunker fuel consumption, for a given vessel with a given cargo, generally depends on distance, speed, and weather conditions. Specifically, to avoid or limit the effect of adverse weather conditions, a vessel can adjust the speed, by anticipating or delaying the departure to avoid or reduce transit time through adverse weather conditions [4]. However, it can be argued that such a change might have a negative impact on the consumption and may hinder the possibility of the liner to meet the schedule requirements.

In this paper, we refer to the problem of modifying speed for bunkering optimization and weather conditions hedging as *the fixed track weather routing problem* (FTWRP) [4]. Specifically, the FTWRP that we consider is the operational problem of determining a vessel's optimal speed under dynamic weather conditions on a fixed track.

Most of the related works in this area do not consider the adjustment of departure time and study only the problem between a pair of ports [5]. [6] develop methodologies for the minimal time routing problem for a vessel in a deterministic weather condition. They use optimal control theory to reveal the optimal policy. [7] studies the deterministic minimal fuel routing problem by applying optimal control theory and dynamic programming methods. The objective is to minimize the fuel consumption while certain safety constraints are met.

Other works related to this study address speed optimization. Most previous works on sailing speed optimization problems do not specifically consider the weather conditions. [8] constructs models to analyse the trade–off between bunkering costs savings through slow steaming and the loss of revenues due to the resulting voyage extension. The models are developed based on different operating scenarios: income generating leg, positioning leg and mixed leg. For each of the models, the optimal speed of a vessel is explicitly determined. [9], [10] study the problem of minimum–cost operation of a fleet of vessels that has to carry a specific amount of cargo between one loading port and one unloading port in a given time period for a specific, fixed contract price. Total fleet operating costs is minimized by choosing the optimal full load and ballast vessel speeds.

We propose a dynamic model in which precise weather forecast information is considered and the optimal speed is dynamically chosen at different time steps along the service by using the calculus of variations. Following [4], we consider the two common practices of fixed track weather routing, adjustment of arrival and departure time and adjustment of speed, within a voyage consisting of multiple ports of call. Without loss of generality, we consider fixed port time (including entry time,unloading time,loading time idle time and exit time). The optimal speed results piecewise constant functions. This has practical value since it is realistic for the vessel's operator to change the speed every few hours.

The remainder of this paper is organized as follows. Section 2 formulates the optimization problem and presents the solving algorithm. Section 3 presents the methodology, while section 5 reports our preliminary numerical outcomes. We conclude and discuss ongoing and future work in Section 5.

II. Problem formulation

During the voyage from an initial port to a destination, the vessel passes through a set of intermediate ports. It loads or unloads cargo, obtains supplies, or undergoes repairs on her service route at each of these ports. Let n be the number of all ports the vessel stops during the voyage. We assign 1 and n to the initial and the final ports, respectively. Each of the intermediate ports is assigned a number $i \in \{1, \ldots, n\}$, according to the order in which they are visited. We denote the distance between port i and the initial port as d_i. We also assume, the total distance of the voyage is D. Mathematically, $0 \le d_i \le D$, where distance of the initial port $d_1 = 0$ and that of the final port $d_n = D$. We consider the voyage begins at time $t = 0$ and the maximum allowable horizon to end the voyage is T. Thus, the position of the vessel at time $t \in [0, T]$ can be presented by a function $x\colon [0, T] \to [0, D]$, such that $x(t = 0) = d_1$. This condition implies at the begining of the voyage the vessel is at port 1. The speed of the vessel at time instance t is denoted by $v(t)$. Following the definition of velocity, we can write $v(t) = \frac{dx(t)}{dt}$. If the vessel is not staying at a port, $v(t)$ is bounded by the minimum possible speed V_{min} and the maximum possible speed V_{max}.

Depending on the velocity $v(t)$ of the vessel, its arrival time t_i at port i varies. But depending on the prior knowledge of the voyage and the traffic, each port i sets soft arriving time window $[t'_i, t''_i]$. It means if the vessel reaches the port i in this window, it will be allowed to dock without any delay or extra charge. But if the vessel reaches the port early or late, it has to pay a certain penalty. The penalty at port i is determined by the function $Q_i\colon [0, T] \to \mathbb{R}^{\ge 0}$. $Q_i(t_i)$ would be zero if $t_i \in [t'_i, t''_i]$ i.e, the vessel reaches during the soft arrival window. It would be decreasing for $t_i \le t'_i$ and increasing for $t_i \ge t''_i$. This intuitively means the penalty would aggravate with the amount of violation. We assume that $Q_i(t_i)$ is convex and differentiable within $[0, t'_i]$ and $[t''_i, T]$.

Let us assume the vessel spends a fixed amount of time e_i at port i. This is called the port time of the vessel at port i. The port time includes entry time, unloading time, loading time, idle time and exit time of the vessel. Hence, the departure time $\bar{t}_i$ at port i can be given by $t_i + e_i$. As the vessel stays at the port during this interval, $v(t) = 0$ for $t_i \le t \le \bar{t}_i$ at each port i. The bunker price at port i is given by c_i and we assume that the vessel will bunker to its full capacity at each port.

Bunker consumption rate is related to the speed of the vessel and weather conditions. For example, ocean wave may cause a percentage speed loss of vessel and the vessel will consume more fuel to maintain her speed. Let $\phi\colon [0, D] \times [0, T] \to \mathbb{R}$ be a function presenting the impact of weather conditions on the bunker consumption rate. We define bunker consumption rate as $F(\phi(x, t), v)$ and assume that F is twice differentiable, convex and strictly increasing on v, when $v \in [V_{min}, V_{max}]$. Therefore the total cost of travelling from port 1 to n is

$$J(v) = \sum_{i=2}^{n} c_i \int_{\bar{t}_{i-1}}^{t_i} F(\phi(x, t), v)dt + \sum_{i=1}^{n} Q_i(t_i). \quad (1)$$

Hence, the problem can be stated mathematically as

$$v* = \underset{v(t)}{\operatorname{argmin}}\, J(v)$$

such that,

$$x(t) = \int_0^t v(\tau)d\tau, 0 \le t \le T, \quad (2)$$

$$x(t) = d_i, \quad t_i \le t \le \bar{t}_i, \quad 1 \le i \le n, \quad (3)$$

$$V_{min} \le v(t) \le V_{max}, \quad \bar{t}_i < t < t_{i+1}, \quad 1 \le i < n. \quad (4)$$

Constraints 2 define the relationship between vessel position x and speed v. Constraints 3 imply that t_i is the arrival time at port i and the position of the vessel is unchanged during its stay at the port. Constraints 4 ensure the speed of the vessel is bounded. Once a feasible $v(t)$ is obtained, $x(t)$, t_i, $\bar{t}_i$ will be uniquely determined by integrating $v(t)$.

We reduce Constraints 4 by incorporating them in Equation 1. Thus, we introduce Lagrange multipliers $\boldsymbol{\lambda}(t) = \binom{\lambda_{max}(t)}{\lambda_{min}(t)}$ and use Karush-Kuhn-Tucker conditions to derive a modified loss function

$$\hat{L}(t, v, x, \boldsymbol{\lambda}) = F(\phi(x,t), v) + \boldsymbol{\lambda}^T \cdot \begin{pmatrix} v - V_{max} \\ V_{min} - v \end{pmatrix}.$$

Hence, we rephrase the problem as

$$\begin{aligned} v* &= \operatorname*{argmin}_{v(t)} J(t, v, x, \boldsymbol{\lambda}) \\ &= \sum_{i=2}^{n} c_i \int_{\bar{t}_{i-1}}^{t_i} \hat{L}(t, v(t), x(t), \boldsymbol{\lambda}(t))dt + \sum_{i=1}^{n} Q_i(t_i) \quad \textbf{(D)} \end{aligned}$$

such that,

$$\begin{gathered} \lambda_{max} \frac{\partial \hat{L}}{\partial \lambda_{max}} = \lambda_{min} \frac{\partial \hat{L}}{\partial \lambda_{min}} \quad \bar{t}_i < t < t_{i+1}, \\ \boldsymbol{\lambda}(t) \geq 0, \quad 0 \leq t \leq T \end{gathered}$$

III. Methodology

In this section, we look for the existence of an optimal solution of the problem and then develop the tools and structure to solve the problem formulated in Section II. We discretize the space and time of the voyage into rectangular patches and accordingly discretize the trajectory of the vessel using Weierstrass-Erdmann condition. This, in turn, gives us governing equations to obtain optimal velocity profile of the vessel. Following this, we construct the algorithm to find unique and optimal velocity.

A. Existence of optimal velocity profile

Before solving the problem, we look for the existence of the solution as it, in turn, guides our method to reach it. In the calculus of variations literature, the optimal velocity profile $v^*(t)$ is called an extremal function or extremal. Let $x^*(t)$ be the associated extremal curve of the vessel with speed $v^*(t)$.

Theorem 1. *Let $\mathcal{W}$ denote the set of function x satisfying* (2)–(4). *There exists $v^* \in \mathcal{W}$ such that $J(v^*) \leq J(v)$ for all $v \in \mathcal{W}$.*

This theorem confirms the existence of the optimal velocity profile and validates the need for an algorithm to find it.

B. Discretization using Weierstrass-Erdmann condition

In this subsection, we propose a discretization of time and space of weather conditions. It is reasonable to incorporate such discretized structure as the weather forecast is updated every a few hours and it also has a space resolution, like a few nautical mile. Using this discretization, we also avoid computing the differentials on the predicted weather conditions, within which errors are expected. Although the solution obtained after the discretization gives the vessel speed at any time instance, later we will show that the obtained solution is a piecewise constant function of time and it is practical for the vessel operator to change the vessel speed every a few hours. Standing on this reasoning, we discretize the space-time as

$$\begin{aligned} 0 = D^0 \leq D^1 \leq \cdots \leq D^{N_d} = D, \\ 0 = T^0 \leq T^1 \leq \cdots \leq T^{N_t} = T. \end{aligned}$$

Without loss of generality, we can assume that for each port of call i, $d_i = D^k$ for some k. Thus, we discretize the whole space and time in rectangular patches $R_{p,q} = [D^p, D^{p+1}) \times [T^q, T^{q+1})$. This discretization is illustrated in Figure 1.We assume that in each $R_{p,q}$, the weather condition remains invariant i.e, ϕ is a constant function in each rectangle $R_{p,q}$. Thus, $F(\phi(x,t), v)$ is same for all $(x,t) \in R_{p,q}$ for a given v. Since in each $R_{p,q}$, $\hat{L}$ is also a function of $v, \boldsymbol{\lambda}$ only, thus $\hat{L}_{p,q}(v, \boldsymbol{\lambda}) = \hat{L}(t, v, x, \boldsymbol{\lambda})$.

If the extremal velocity profile is $v^*(t)$ and corresponding trajectory of the vessel is $x^*(t)$, Theorem 1 shows that in each rectangle R_{p^k,q^k} the optimal speed $v^*(t)$ is constant and the extremal curve $x^*(t)$ is a line segment. The slope of $x^*(t)$ with respect to time presents the vessel's speed in each R_{p^k,q^k}.

Lemma 1. *Suppose that extremal curve $x^*(t)$ will pass through a sequence of rectangles (R_{p^k,q^k}). Then $v^*(t)$ is constant in each R_{p^k,q^k}.*

As the optimal velocity profile is constant in each of the rectangles, the problem is reduced to a piecewise calculation of respective constants in each of the patches. While calculating the optimal solution by pieces, we come across two scenarios. In one case, the vessel is leaving one rectangle and entering another without stopping. In the other case, it enters and leaves a port of call. Using Weierstrass-Erdmann condition, we derive two necessary conditions of extremal curve in Theorems 2 and 3 for these two cases respectively. Theorem 2 and Corollary 1 state that the vessel's speed in $R_{p^{k+1},q^{k+1}}$ is uniquely determined by its leaving speed, leaving time and leaving position in R_{p^k,q^k}.

Theorem 2. *Let*

$$\begin{aligned} K_1^k(v, \boldsymbol{\lambda}) &= \hat{L}_{p^k,q^k}(v, \boldsymbol{\lambda}) - \frac{\partial \hat{L}_{p^k,q^k}}{\partial v}(v, \boldsymbol{\lambda}) \times v \\ K_2^k(v, \boldsymbol{\lambda}) &= \frac{\partial \hat{L}_{p^k,q^k}}{\partial v}(v, \boldsymbol{\lambda}). \end{aligned}$$

Suppose that extremal curve will pass through a sequence of rectangles (R_{p^k,q^k}) for $1 \leq k \leq K$. Let t^k be the time when the extremal curve enters region R_{p_k,q_k}. For $2 \leq k \leq K$, $v^*(t)$ is an extremal only if the following corner conditions is satisfied:

- *if $p^{k-1} + 1 = p^k$ then*
$$K_1^{k-1}(v^*(t^k-), \boldsymbol{\lambda}^*(t^k-)) - K_1^k(v^*(t^k+), \boldsymbol{\lambda}^*(t^k+)) = 0. \tag{5}$$
- *if $q^{k-1} + 1 = q^k$ then*
$$K_2^{k-1}(v^*(t^k-), \boldsymbol{\lambda}^*(t^k-)) - K_2^k(v^*(t^k+), \boldsymbol{\lambda}^*(t^k+)) = 0. \tag{6}$$

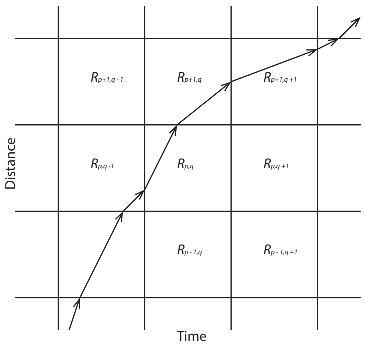

Fig. 1: An illustration of the extremal trajectory.

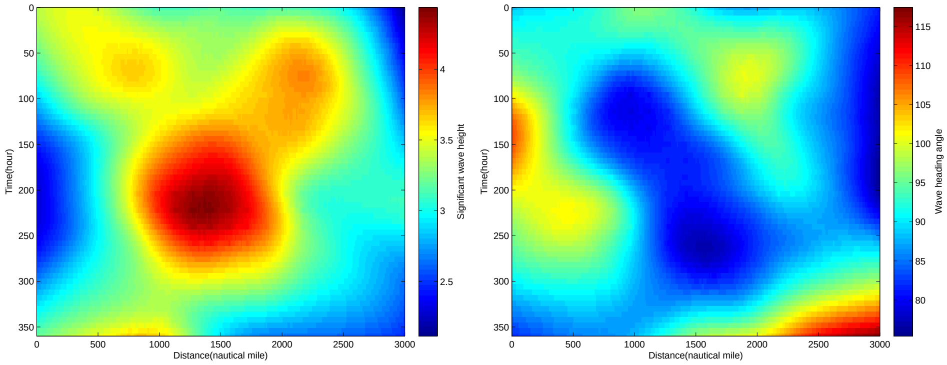

Fig. 2: Randomly generated weather conditions: (left) Significant wave height and (right) Angle between the vessel's heading and wave direction.

Corollary 1. *For given $(v^*(t^k-), \boldsymbol{\lambda}^*(t^k-))$, there exists an unique $(v^*(t^k+), \boldsymbol{\lambda}^*(t^k+))$ satisfying the corner conditions given in 2.*

For $V_{min} < v^*(t) < V_{max}$ and $\boldsymbol{\lambda}^*(t) = \mathbf{0}$, the necessary conditions of Equations 5 and 6 for optimality are known as the Weierstrass–Erdmann corner conditions. Theorem 3 and Corollary 2 state that the speed with which a vessel leaves a port of call is uniquely determined by its arrival time and the speed with which it is arriving at that port.

Theorem 3. *Let*

$$K_3(t, v, x, \boldsymbol{\lambda}) = \hat{L}(t, v, x, \boldsymbol{\lambda}) - \frac{\partial \hat{L}}{\partial v}(t, v, x, \boldsymbol{\lambda}) v(t)$$

At the first port of call, $v^*(t)$ is an extremal only if the following is satisfied:

$$q_1'(t_1) - c_2 K_3(\bar{t}_1+, v^*, x^*, \boldsymbol{\lambda}^*) = 0. \tag{7}$$

For the port of call i with $2 \le i \le n-1$, $v^*(t)$ is an extremal only if the following is satisfied:

$$q_i'(t_i) + c_i K_3(t_i-, v^*, x^*, \boldsymbol{\lambda}^*) - c_{i+1} K_3(\bar{t}_i+, v^*, x^*, \boldsymbol{\lambda}^*) = 0. \tag{8}$$

Corollary 2. *For given departure time $\bar{t}_1$ of the first port of call, there exists unique $v(\bar{t}_1+), \boldsymbol{\lambda}(\bar{t}_1+)$ such that Equation 7 is satisfied. For $2 \le i \le n$, for given $t_i, v(t_i-), \boldsymbol{\lambda}(t_i-)$, there exists unique $v(\bar{t}_i+), \boldsymbol{\lambda}(\bar{t}_i+)$ such that equation (8) is satisfied.*

Based on these propositions, we provide an algorithm to obtain the local optimal solution for velocity profile in Section III-C.

C. Algorithm

In this subsection, we propose and summarize the algorithm to solve our problem. For the vessel speed $v(t)$ to be optimal, the associated departure speed v at the first port along with the Lagrange multiplier, $\boldsymbol{\lambda}$ should satisfy Equation 7. Therefore for a given departure time $\bar{t}_1$ at the first port, we obtain unique (by Corollary 2) departure speed and multiplier, say $v_1(\bar{t}_1)$ and $\boldsymbol{\lambda}(\bar{t}_1)$. In each rectangle $R_{p,q}$, the optimal speed $v(t)$ is a constant function of time (by Lemma 1) and the behaviour of the optimal speed $v(t)$ on the boundaries of $R_{p,q}$ is uniquely determined by Theorem 2. For each port of call, given the arrival speed and time, the departure speed is uniquely determined (by Theorem 3). Thus, given the departure time at the first port $\bar{t}_1$, we can obtain the unique solution $v(t, \bar{t}_1)$ satisfying the necessary optimality conditions. Therefore we need to find the departure time of the first port of call $\bar{t}_1$ such that $J(v(t, \bar{t}_1))$ is minimized, i.e. $\min_{\bar{t}_1 \in [t_{min}, t_{max}]} J(v(t, \bar{t}_1))$. This reduced problem can be solved locally by using derivative-free methods like search and interpolation. We summarize this aforementioned technique to obtain locally optimal velocity profile in Algorithm 1.

D. An approach to global extension

In order to extend our proposed algorithm to obtain a global optimal, we can derive regularity constraints on the function and apply branch and bound techniques to leverage it. If $J(v(t, \bar{t}_1))$ is a Lipschitz continuous function [11] of $\bar{t}_1$ with some Lipschitz constant M, global information in the form of the Lipschitz constant can be used to compute lower bounds on $J(v(t, \bar{t}_1))$ over each subregion. This approach of accodomating globality in a local solution using Lipschitz constant has been studied in the literature [12]. Let δ be a small enough time interval such that $M\delta \le \epsilon$. Then by definition of continuity, for any η such that $0 < \eta \le \delta$

$$|J(v(t, \bar{t}_1)) - J(v(t, \bar{t}_1 + \eta))| \le \epsilon.$$

Hence the optimal value of the following problem

$$\min_i \{J(v(t, \bar{t}_1) \mid \bar{t}_1 = t_{min} + k\delta \le t_{max}, \quad k \in \mathbb{Z}_{\ge 0}\}.$$

turns out to be a global ϵ–optimal solution. Since in our particular setting M may be extremely large and it is time consuming the find a globally optimal solution, in the experimental section we only use the local optimal solution.

IV. Experimental analysis

A. General set-up

In order to validate the performance of our algorithm, we build up a setting where a vessel's voyage consists of 4, not necessarily different, ports. The distance of each

Algorithm 1 Compute $v(t, \bar{t}_1)$

procedure $v(t, \bar{t}_1)$
  Determine departure speed v^1 and adjoint parameter $\boldsymbol{\lambda}^1$ from $\bar{t}_1$ by equation (7).
  $t^1 \leftarrow \bar{t}_1$, $x(t^1) \leftarrow d_1$, $v(t^1) \leftarrow v^1$, $\boldsymbol{\lambda}(t^1) \leftarrow \boldsymbol{\lambda}^1$
  $p \leftarrow \min\{j \mid D^j \geq x(t^1), 0 \leq j \leq N^d\}$, $q \leftarrow \min\{j \mid T^j \geq t^1, 0 \leq j \leq N^t\}$
  for $i \leftarrow 2 : n$ **do**
    $k \leftarrow 1$
    while $x(t^k) < d_i$ **do**
      if $v^k(T^{q+1} - t^k) + x(t^k) < \min\{D^{p+1}, d_i\}$ **then**
        $t^{k+1} = T^{q+1}$
        $q \leftarrow q + 1$
        Determine $v^{k+1}, \boldsymbol{\lambda}^{k+1}$ from t^{k+1}, v^k and $\boldsymbol{\lambda}^k$ by equation (6).
      else
        $t^{k+1} \leftarrow \frac{1}{v^k}(\min\{D^{p+1}, d_i\} - x(t^k)) + t^k$
        $p \leftarrow p + 1$
        Determine $v^{k+1}, \boldsymbol{\lambda}^{k+1}$ from t^{k+1}, v^k and $\boldsymbol{\lambda}^k$ by equation (5).
      end if
      for all $t \in [t^k, t^{k+1})$ **do**
        $x(t) \leftarrow v^k(t - t^k) + x(t^k)$, $v(t) \leftarrow v^k$, $\boldsymbol{\lambda}(t) \leftarrow \boldsymbol{\lambda}^k$
      end for
      $k \leftarrow k + 1$
    end while
    $\bar{t}_i \leftarrow t^k + p_i$
    for all $t \in [t^k, \bar{t}_i)$ **do**
      $x(t) \leftarrow d_i$, $v(t) \leftarrow 0$
    end for
    Determine v^1 and $\boldsymbol{\lambda}^1$ from t^k, $\bar{t}_i$, v^k and $\boldsymbol{\lambda}^k$ by equation (8).
    $p \leftarrow \min\{j \mid D^j \geq x(\bar{t}_i), 0 \leq j \leq N^d\}$, $q \leftarrow \min\{j \mid T^j \geq \bar{t}_i, 0 \leq j \leq N^t\}$
  end for
  return $v(t)$.
end procedure

leg is 1000nm. The departure window at the first port is $[27h, 30h]$. The arrival window at the rest of the three ports are $[114h, 117h]$, $[207h, 210h]$, $[306h, 309h]$ respectively. We do not consider longer voyage because the meteorological prediction of weather can be unreliable when looking at long time periods ahead. Beside this, we assume that the bunker price at all ports equals to one as we want to concentrate on the weather effect.

We can approximate the bunker consumption rate from the statistical method by [13]. This method was developed through a regression analysis of random model experiments and full-scale data. The engine's brake power P_B is given by

$$P_B(V) = \frac{R_{total}(V) \times V}{\eta_{GB}\eta_R\eta_O\eta_S\frac{1-t_d}{1-w(V)}},$$

TABLE II: Main particulars of the vessel

Length on waterline	180m
Length between perpendiculars	175m
Breadth moulded	25.4m
Average draught moulded	9.5m
Displacement volume moulded	21180m^3
Transverse build area	20m
Center of bulb area above keel line	4m
Midship section coefficient	0.98
Waterplane area coefficient	0.750
Transom area	16m^2
Wetted area appendages	50m^2
Stern shape parameter	10
Propeller diameter	8m
Number of propeller blades	4
Auxiliary engine power	750kW
Auxiliary engine load factor in maneuver	50%
Main engine specific fuel consumption coefficient	190g/kWh
Auxiliary engine specific fuel consumption coefficient	215g/kWh

where R_{total} is total resistance for vessel speed V, η_R is the relative-rotative efficiency, η_O is the nominal efficiency, η_S is the shafting efficiency, η_{GB} is the gearbox efficiency, $w(V)$ is the wake fraction for vessel speed V and, t_d is the thrust deduction. These parameters can be computed using the method provided in [13]. The bunker consumption rate can be obtained by using

$$F(V) = C_{ME}P_B(V) + C_{AE}P_{AE}L_{AE}$$

where C_{ME} and C_{AE} are specific fuel consumption coefficients, P_{AE} is the auxiliary engine power and L_{AE} is the auxiliary engine load factor for the vessel in the maneuver.

In our specific setting as shown in Table IV-A, the relationship between vessel's speed and bunker consumption rate can be approximated by the polynomial $0.0006V^3 - 0.2291V + 2.3294$. A reliable prediction of attainable vessel's speed at actual seas is essential. The effect of waves has an influence on the speed loss of the vessel. We use the method provided in [14] to estimate the speed loss. Let $\phi(h, \theta)$ present percentage of speed loss for the significant wave height h and the wave heading angle θ. As shown in [14], $\phi(h, \theta)$ can be given by

$$\phi(h, \theta) = 1 + \mu(h, \theta) \times (0.0284 \times h^{1/3} + 0.0054 \times h^{13/6}),$$

where,

$$\mu(h, \theta) = \begin{cases} 1 & \theta \leq 30 \\ \frac{1.7 - 0.03 \times (4.0632 \times h^{1/3} - 4)^2}{2}, & 30 < \theta \leq 60 \\ \frac{0.9 - 0.03 \times (4.0632 \times h^{1/3} - 6)^2}{2}, & 60 < \theta \leq 150 \\ \frac{1.7 - 0.03 \times (4.0632 \times h^{1/3} - 8)^2}{2}. & 150 < \theta \end{cases}$$

If the effect of waves is taken into account, the resulting effective speed $v_e = \phi(h, \theta)v_{inst}$. v_{inst} is the instantaneous speed, derived on the basis of two consecutive GPS observations. The resulting effective speed v_e is used to calculate bunker consumption rate.

In this experiment, the penalty cost for violation of the arriving time window is given by

$$\sum_{i=1}^{n} c'_i \left([t_i - t''_i]^+ + 0.5[t'_i - t_i]^+\right)$$

where t_i is the arrival time of the vessel at the port i, c_i' denotes the penalty per hour per tons of bunker and,

$$[t]^+ = \begin{cases} t & t > 0, \\ 0 & t \leq 0. \end{cases}$$

The randomly generated weather forecast, which is shown in Figure 2,starts at time 0h and ends at time 360h. The resolution is 6 hours and 10 nm. In this randomly generated weather forecast, the mean of significant wave height is 3 meters and the standard deviation is 0.5 meters.

B. Performance analysis

In order to compute the optimal vessel speed between two ports, we need to find the solution satisfying the necessary optimality for given initial speed, time and position. Our proposed method to achieve this is summarized in Algorithm 1. In this paper, we use an algorithm based on the golden section search and parabolic interpolation [15], [16] to obtain the local solution. This algorithm is provided in MATLAB [17] as the **fminbnd** function.

Another possible method to solve this problem is using the initial value problem (IVP) induced by the Euler-Lagrange equation

$$\frac{\partial}{\partial x}\hat{L} - \frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\partial}{\partial v}\hat{L} = 0.$$

To numerically solve the IVP, we need to compute the differential of $\hat{L}$. To obtain the differentiable $\hat{L}$, we interpolate the discrete weather forecast information using the [17] function **interp2** with interpolation method **cubic**.

Within different randomly generated weather conditions, we randomly choose 1000 departure times and 1000 departure speed. With each pair of departure time and departure speed, we compute Algorithm 1 and the IVP until the distance travelled equals to 1000nm. The computation is carried out on a desktop with two Intel© Xeon© E5–2609 processors and 32GB installed physical memory. The mean computation time of our algorithm is 0.0197s with standard deviation 0.0062s while the mean computation time of IVP is 0.5298s with standard deviation 0.1071s. Since in real-life scenario weather and forecasts change in certain intervals, Algorithm 1 will be repeatedly invoked to solve the stochastic problem. Thus, the decreased computation time of our algorithm makes it significantly efficient. Besides this, the standard deviation of computation time for the proposed algorithm is significantly less that proves its stability over IVP.

Another problem we need to consider is that errors are to be expected in the predicted weather conditions. To solve IVP, we need to compute $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial t}$. When errors are expected in the value of function ϕ, these errors may become expanded to an unacceptable level when differentials on ϕ are computed. But for Algorithm 1, we do not need to compute such differentials.

V. Conclusion

In this paper, we study the operational problem of vessel time management and speed optimization under dynamic weather conditions. We model the minimization of cost using the calculus of variations with the weather as a time-varying model.

The model and the solution we propose are generic enough to be applicable to most bunker consumption rate model and weather conditions models. We validate our proposal with a case in which, for the sake of clarity, we have borrowed the model of bunker consumption of [13], [14], [18] and we have used a simple model of weather conditions.

We believe that our model is practical, efficient and effective enough to be implemented and to give an operational level decision support to lower the overall bunker cost.

However, several issues remain to be studied. First, we need to extend the weather model to consider uncertainty in the prediction. This is particularly important to make the algorithm applicable in the case of long intercontinental routes when the prediction of the weather is subject to remarkable variance which need to be considered to make the solution more robust.

We are currently considering other navigational and safety constraints such as the ability of a vessel to change speed and the avoidance of extreme weather. We are also currently considering the relevance of bunker price forecasting models to take into account the uncertainty of bunker prices.

References

[1] UNEP, *IMO GHG Study 2009*. IMO, 2009.

[2] D. Ronen, "The effect of oil price on containership speed and fleet size," *Journal of the Operational Research Society*, vol. 62, no. 1, pp. 211–216, Jan. 2010.

[3] T. E. Notteboom and B. Vernimmen, "The effect of high fuel costs on liner service configuration in container shipping," *Journal of Transport Geography*, vol. 17, no. 5, pp. 325 – 337, 2009.

[4] L. Nathaniel Bowditch, *The American Practical Navigator*, american practical navigator ed. American Practical Navigator, 2011.

[5] T. Thornton, "A review of weather routeing of sailboats," *Journal of Navigation*, vol. 46, pp. 113–129, 1 1993.

[6] N. A. Papadakis and A. N. Perakis, "Deterministic minimal time vessel routing," *Operations Research*, vol. 38, no. 3, pp. 426–438, 1990.

[7] K. Avgouleas, "Optimal ship routing," Thesis, Massachusetts Institute of Technology, 2008.

[8] D. Ronen, "The effect of oil price on the optimal speed of ships," *The Journal of the Operational Research Society*, vol. 33, no. 11, pp. pp. 1035–1040, 1982.

[9] A. N. Perakis and N. Papadakis, "Fleet deployment optimization models. part 1," *Maritime Policy & Management*, vol. 14, no. 2, pp. 127–144, 1987.

[10] ——, "Fleet deployment optimization models.part 2," *Maritime Policy & Management*, vol. 14, no. 2, pp. 145–155, 1987.

[11] J. Heinonen, *Lectures on Lipschitz analysis*, 2005.

[12] A. Neumaier, "Complete search in continuous global optimization and constraint satisfaction," *Acta Numerica*, vol. 13, pp. 271–369, 5 2004.

[13] J. Holtrop and G. Mennen, "An approximate power prediction method," *International Shipbuilding Progress*, vol. 29, no. 335, pp. 166–170, 1982.

[14] J.-P. Jalkanen, A. Brink, J. Kalli, H. Pettersson, J. Kukkonen, and T. Stipa, "A modelling system for the exhaust emissions of marine traffic and its application in the baltic sea area," *Atmospheric Chemistry and Physics*, vol. 9, no. 23, pp. 9209–9223, 2009.

[15] G. E. Forsythe, M. A. Malcolm, and C. B. Moler, *Computer Methods for Mathematical Computations*. Englewood Cliffs, NJ: Prentice-Hall, 1977.

[16] R. P. Brent, *Algorithms for Minimization Without Derivatives*. Englewood Cliffs, New Jersery: Prentice-Hall, Inc., 1973.

[17] MATLAB, *version 8.2.0.348 (R2013b)*. Natick, Massachusetts: The MathWorks Inc., 2013.

[18] R. L. Townsin and Y. J. Kwon, "Estimating the influence of weather on ship performance," *RINA Trans*, vol. 135, 1993.